Law of Large Numbers for products of random matrices with coefficients in the max-plus semi-ring.

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Abstract

We analyze the asymptotic behavior of random variables $x(n,x_0)$ defined by $x(0,x_0)=x_0$ and $x(n+1,x_0)=A(n)x(n,x_0)$, where $\left(A(n)\right)_{n\in\mathbb{N}}$ is a stationary and ergodic sequence of random matrices with entries in the semi-ring $\mathbb{R}\cup\{-\infty\}$ whose addition is the max and whose multiplication is $+$.

Such sequences modelize a large class of discrete event systems, among which timed event graphs, 1-bounded Petri nets, some queuing networks, train or computer networks. We give necessary conditions for $\left(\frac{1}{n}x(n,x_0)\right)_{n\in\mathbb{N}}$ to converge almost surely. Then, we prove a general scheme to give partial converse theorems. When $\max_{A_{ij}(0)\neq-\infty}|A_{ij}(0)|$ is integrable, it allows us:

- to give a necessary and sufficient condition for the convergence of $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ when the sequence $\left(A(n)\right)_{n\in\mathbb{N}}$ is i.i.d.,
- to prove that, if $\left(A(n)\right)_{n\in\mathbb{N}}$ satisfy a condition of reinforced ergodicity and a condition of fixed structure (i.e. $\mathbb{P}\left(A_{ij}(0)=-\infty\right)\in\{0,1\}$), then $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ converges almost-surely,
- and to reprove the convergence of $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ if the diagonal entries are never $-\infty$.

1 Introduction

1.1 Model

We analyze the asymptotic behavior of random variables $x(n,x_0)$ defined by:

$$\left\{\begin{array}{lcl} x(0,x_0) & = & x_0 \\ x(n+1,x_0) & = & A(n)x(n,x_0), \end{array}\right. \tag{1}$$

where $(A(n))_{n\in\mathbb{N}}$ is a stationary and ergodic sequence of random matrices with entries in the semi-ring $\mathbb{R}\cup\{-\infty\}$ whose addition is the max and whose multiplication is $+$.

We also define the product of matrices $A^n := A(n-1)A(n-2)\cdots A(0)$ such that $x(n,x_0) = A^n x_0$ and, if the sequence has indices in $\mathbb{Z}$, which is possible up to a change of probability space, $A^{-n} := A(-1)\cdots A(-n)$ and $y(n,x_0) := A^{-n}x_0$.

On the coefficients, Relation (1) reads

$$x_i(n+1,x_0) = \max_j \left(A_{ij}(n) + x_j(n,x_0)\right),$$

and the product of matrices is defined by

$$(A^n)_{ij} = \max_{i_0=j, i_n=i} \sum_{l=0}^{n-1} A_{i_{l+1}i_l}(l). \tag{2}$$

In most cases, we assume that $A(n)$ never has a line of $-\infty$, which is a necessary and sufficient condition for $x(n,x_0)$ to be finite. (Otherwise, some coefficients can be $-\infty$.)

Such sequences modelize a large class of discrete event systems. This class includes some models of operations research like timed event graphs (F. Baccelli [Bac92]), 1-bounded Petri nets (S. Gaubert and J. Mairesse [GM99]) and some queuing networks (J. Mairesse [Mai97], B. Heidergott [Hei00]) and many concrete applications. Let us cite job-shops models (G. Cohen et al.[CDQV85]), train networks (H. Braker [Bra93], A. de Kort and B. Heidergott [dKHA03]), computer networks (F. Baccelli [BH00]) or a statistical mechanics model (R. Griffiths [Gri90]). For more details about modelling, the reader is referred to the books by F. Baccelli and al. [BCOQ92] and by B. Heidergott and al. [HOvdW06].

1.2 Law of large numbers

The sequences satisfying Equation (1) have been studied in many papers. Law of large numbers have been proved among others by J.E. Cohen [Coh88], F. Baccelli and Z. Liu [BL92], and more recently by T. Bousch and J. Mairesse [BM03].

If matrix A has no line of $-\infty$, then $x \mapsto Ax$ is 1-Lipschitz for the supremum norm. Therefore, we can assume $x_0 = 0$, and we do it from now on.

T. Bousch and J. Mairesse have proved (cf. [BM03]) that, if $A(0)0$ is integrable, then the sequence $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$ converges almost-surely and in mean. The proof is still true if $\max_{ij} A_{ij}^+(0)$ is integrable and the limit can be $-\infty$.

Therefore, the sequence $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ converges in law. But, it does not necessary converges almost-surely, as illustrated by examples bellow. T. Bousch and J. Mairesse have also proved that, if $\max_{A_{ij}(0)\neq-\infty} |A_{ij}(0)|$ is integrable, then $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ convergences almost-surely if and only if the limit of $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$

is constant. Thanks to their former results, we give necessary conditions for $\left(\frac{1}{n}x(n,x_0)\right)_{n\in\mathbb{N}}$ to converge almost surely under the usual integrability condition.

F. Baccelli has proved (cf. [Bac92]) by induction on the size of the matrices, that $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ converges almost-surely under the following additional hypotheses: each entry of the matrix is either almost-surely $-\infty$, or almost-surely non-negative and the diagonal entries are non-negative (precedence condition). T. Bousch and J. Mairesse have proved (cf. [BM03]) that, if $\max_{A_{ij}(0)\neq-\infty}|A_{ij}(0)|$ is integrable, then the precedence condition is sufficient. Practically, the limit of $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$ is constant if the diagonal coefficients are almost-surely finite. We use the induction method without the additional hypotheses. We obtain a general scheme (Theorem 2.4) to prove that $\left(\frac{1}{n}x(n,x_0)\right)_{n\in\mathbb{N}}$ converges almost surely. When $\max_{A_{ij}(0)\neq-\infty}|A_{ij}(0)|$ is integrable, it allows us:

- to give a necessary and sufficient condition for the convergence of $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ when the sequence $(A(n))_{n\in\mathbb{N}}$ is i.i.d.,
- to prove that, if $(A(n))_{n\in\mathbb{N}}$ satisfy a condition of reinforced ergodicity and a condition of fixed structure (i.e. $\mathbb{P}\left(A_{ij}(0)=-\infty\right)\in\{0,1\}$), then $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ converges almost-surely,
- and to reprove the convergence of $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ if the diagonal entries are never $-\infty$.

In the next section, we will state our results, then we will prove a necessary condition for the convergence of $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ (Theorem 2.3), then the sufficient condition (Theorem 2.4) and finally its three consequences.

2 Presentation of the results

The first result is the following, which directly follows from Kingman's theorem and can be traced back to J.E. Cohen [Coh88]:

Theorem-Definition 2.1 (Maximal Lyapunov exponent).
If $(A(n))_{n\in\mathbb{N}}$ is an ergodic sequence of random matrices with entries in $\mathbb{R}_{\max}$, such that $\max_{ij}A^+_{ij}(0)$ is integrable, then the sequences $\left(\frac{1}{n}\max_i x_i(n,0)\right)_{n\in\mathbb{N}}$ and $\left(\frac{1}{n}\max_i y_i(n,0)\right)_{n\in\mathbb{N}}$ converges almost-surely to the same constant $\gamma\in\mathbb{R}_{\max}$, which is called maximal Lyapunov exponent of $(A(n))_{n\in\mathbb{N}}$.

We denote this constant by $\gamma\left((A(n))_{n\in\mathbb{N}}\right)$, or $\gamma(A)$.

Remark 2.1. The constant $\gamma(A)$ is well-defined even if $(A(n))_{n\in\mathbb{N}}$ has a line of $-\infty$. The variable $\max_i x_i(n,0)$ is equal to $\max_{ij}A^n_{ij}$.

Let us associate to our sequence of random matrices a graph to split the problem. We also set the notations for the rest of the text.

Définition 2.2 (Graph of possible incidences). For every $x \in \mathbb{R}_{\max}^{[1,\cdots,d]}$ and every subset $I \subset [1, \cdots, d]$, we write:

$$x^I := (x_i)_{i \in I}.$$

Let $(A(n))_{n\in\mathbb{N}}$ be a stationary sequence of random matrices with values in $\mathbb{R}_{\max}^{d\times d}$.

i) The graph of possible incidences of $(A(n))_{n\in\mathbb{N}}$, denoted by $\mathcal{G}(A)$, is the directed graph whose nodes are the integers between 1 and d and whose arcs are the pairs (i,j) such that $\mathbb{P}(A_{ij}(0) \neq -\infty) > 0$.

ii) We denote by $c_1, \cdots, c_K$ the strongly connected components of $\mathcal{G}(A)$. In the sequel, we just say components of $\mathcal{G}(A)$.

   To each component c_m, we associate the following elements:

$$A^{(m)}(n) := (A_{ij}(n))_{i,j\in c_m},\ \gamma^{(m)} := \gamma(A^{(m)}),$$

$$x^{(m)}(n, x_0) := (A^{(m)})^n (x_0)^{c_m} \text{ and } y^{(m)}(n, x_0) := (A^{(m)})^{-n} (x_0)^{c_m}$$

iii) A component c_l is reachable from a component c_m, if $m = l$ or if there exists a path on $\mathcal{G}(A)$ from a node in c_m to a node in c_l. In this case, we write $m \to l$.

   To each component c_m, we associate the following elements:

$$E_m := \{l \in [1, \cdots, K] | m \to l\},\ \gamma^{[m]} := \max_{l\in E_m} \gamma^{(l)},$$

$$F_m := \bigcup_{l\in E_m} c_l,\ A^{[m]}(n) := (A_{ij}(n))_{i,j\in F_m}$$

$$x^{[m]}(n, x_0) := \left(A^{[m]}\right)^n (x_0)^{F_m} \text{ and } y^{[m]}(n, x_0) := \left(A^{[m]}\right)^{-n} (x_0)^{F_m}.$$

iv) A component c_m is final (or source, in the terminology of discrete event systems) if $E_m = \{m\}$, that is if, for every $l \in [1, \cdots, K]$, we have:

$$m \to l \Rightarrow l = m.$$

   It is initial if, for every $l \in [1, \cdots, K]$, we have:

$$l \to m \Rightarrow l = m.$$

   A component is said to be trivial, if it has only one node i and $\mathbb{P}(A_{ii}(1) \neq -\infty) = 0$.

v) To each component c_m, we associate the following sets:

$$G_m := \{l \in E_m | \exists p \in [1, \cdots, K], m \to l \to p, \gamma^{(p)} = \gamma^{[m]}\},$$

$$H_m := \bigcup_{l \in G_m} c_l \ , \ A^{\{m\}}(n) := (A_{ij}(n))_{i,j \in H_l}$$

$$x^{\{m\}}(n, x_0) := \left(A^{\{m\}}\right)^n (x_0)^{H_m} \text{ and } y^{\{m\}}(n, x_0) := \left(A^{\{m\}}\right)^{-n} (x_0)^{H_m}.$$

vi) A component c_m is called dominating if $G_m = \{m\}$, that is if for every $l \in E_m \backslash \{m\}$, we have: $\gamma^{(m)} > \gamma^{(l)}$.

Remark 2.2 *(Paths on $\mathcal{G}(A)$).* Equation (2) can be read as 'A_{ij}^n is the maximum of the weights of paths from i to j with length n on $\mathcal{G}(A)$, the weight of the k^{th} arc being given by $A(-k)$'. Thus $y_i(n, 0)$ is the maximum of the weights of paths on $\mathcal{G}(A)$ with initial node i and length n. The coefficients $y_i^{(m)}(n, 0)$ and $y_i^{\{m\}}(n, 0)$ are the maximum of the weights of paths on the subgraph of $\mathcal{G}(A)$ with nodes in c_m and H_m respectively.

Consequently $\gamma^{(m)}$ is the average maximal weight of path on c_m

The first new result is a necessary condition for $x(n, X_0)$ to satisfy a strong law of large numbers:

Theorem 2.3. *Let $(A(n))_{n \in \mathbb{N}}$ be a stationary and ergodic sequence of random matrices with values in $\mathbb{R}_{\max}^{d \times d}$ and no line of $-\infty$, such that $\max_{ij} A_{ij}^+(0)$ is integrable.*

If the limit of $\left(\frac{1}{n} y(n, 0)\right)_{n \in \mathbb{N}}$ is deterministic, then it is given by:

$$\forall m \in [1, K], \lim_n \frac{1}{n} y^{c_m}(n, 0) = \gamma^{[m]} \mathbf{1} \text{ a.s.,} \tag{3}$$

where $\mathbf{1}$ is the vector whose coordinates are all 1.
That being the case, for every component c_m of $\mathcal{G}(A)$, $A^{\{m\}}(0)$ has no line of $-\infty$.

If $\left(\frac{1}{n} x(n, 0)\right)_{n \in \mathbb{N}}$ converges almost-surely, then its limit is deterministic and is equal to that of $\left(\frac{1}{n} y(n, 0)\right)_{n \in \mathbb{N}}$, that is we have:

$$\forall m \in [1, K], \lim_n \frac{1}{n} x^{c_m}(n, 0) = \gamma^{[m]} \mathbf{1} \text{ a.s.,} \tag{4}$$

The following theorem gives a scheme to prove converse theorems:

Theorem 2.4. *Let $(A(n))_{n \in \mathbb{N}}$ be an ergodic sequence of random matrices with values in $\mathbb{R}_{\max}^{d \times d}$ that satisfy the three following hypotheses:*

1. *for every component c_m of $\mathcal{G}(A)$, $A^{\{m\}}(0)$ has no line of $-\infty$.*

2. *for every dominating component c_m of $\mathcal{G}(A)$, $\lim_n \frac{1}{n} y^{(m)}(n,0) = \gamma^{(m)}\mathbf{1}$ a.s..*

3. *for every subsets I and J of $[1,\cdots,d]$, such that random matrices $\tilde{A}(n) = \left(A_{ij}(n)\right)_{i,j\in I\cup J}$ has no line of $-\infty$ and split along I and J following the equation*
$$\tilde{A}(n) =: \begin{pmatrix} B(n) & D(n) \\ -\infty & C(n) \end{pmatrix}, \tag{5}$$
such that $\mathcal{G}(B)$ is strongly connected and $D(n)$ is not almost surely $(-\infty)^{I\times J}$, we have:
$$\mathbb{P}\left(\{\exists i \in I, \forall n \in \mathbb{N}, \left(B(-1)\cdots B(-n)D(-n-1)0\right)_i = -\infty\}\right) = 0. \tag{6}$$

Then the limit of $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$ is given by Equation (3).

If Hypothesis 1. is strengthened by demanding that $A^{\{m\}}(0)0$ is integrable, then the sequence $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ converges almost-surely and its limit is given by Equation (4).

Remark 2.3 *(Paths on $\mathcal{G}(A)$, continued).* Let us interpret the three hypotheses with the paths on $\mathcal{G}(A)$.

1. The hypothesis on $A^{\{m\}}(0)$ means that, whatever the initial condition $i \in c_m$, there is always a path beginning in i and staying in H_m.

2. The hypothesis on dominating component means that, whatever the initial condition i in dominating component c_m, there is always a path beginning in i with average weight $\gamma^{(m)}$. It is necessary, as can be shown by an method analogeous to that of [Bac92].

3. We will use the last hypothesis with $\tilde{A}(n) = A^{\{m\}}(n)$, $B(n) = A^{(m)}(n)$. It means there is a path from $i \in c_m$, to $H_m \backslash c_m$. Once we know that the limit of $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$ is given by Equation (3) this hypothesis is obviously necessary when $\gamma^{(m)} < \gamma^{[m]}$.

Thanks to hypotheses 1. and 3., for every component c_m and every node $i \in c_m$, there is always a path beginning in i reaching a dominating component c_k with Lyapunov exponent $\gamma^{(k)} = \gamma^{[m]}$ and staying in that component. Thanks to Hypothesis 2., this paths has $\gamma^{[m]}$ as average weight.

It remains to prove that there is no path in F_m with average weight strictly greater than $\gamma^{[m]}$ and goes from Equation (3) to Equation (4). This is possible thanks to theorem 3.2, from [Bac92] and theorem 3.4, from [Vin97] respectively.

The three announced results follow from this scheme:

Theorem 2.5 (Independent case). *If $(A(n))_{n\in\mathbb{N}}$ is a sequence of i.i.d. random matrices with values in $\mathbb{R}_{\max}^{d\times d}$ and no line of $-\infty$, such that $\max_{A_{ij}(0)\neq-\infty}|A_{ij}(0)|$ is integrable, then the sequence $\left(\frac{1}{n}x(n,0)\right)$ converges almost-surely*[1] *if and only if for every component c_m, $A^{\{m\}}$ has no line of $-\infty$. That being the case the limit is given by Equation (4).*

Theorem 2.6 (Fixed structure case). *If $(\Omega,\theta,\mathbb{P})$ is a measurable dynamical system, and $A:\Omega\to\mathbb{R}_{\max}^{d\times d}$ is a random matrix with no line of $-\infty$, such that:*

1. *for every i,j, A_{ij} is integrable or almost-surely $-\infty$,*
2. *for every $k\leq d$, θ^k is ergodic,*

then, the sequence $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ associated to the sequence $(A(n))_{n\in\mathbb{N}}=(A\circ\theta^n)_{n\in\mathbb{N}}$ converges almost-surely[1] *and its limit is given by Equation (4).*

Theorem 2.7 (Precedence case). *If $(A(n))_{n\in\mathbb{N}}$ is a stationary and ergodic sequence of random matrices with values in $\mathbb{R}_{\max}^{d\times d}$ such that $\max_{A_{ij}(0)\neq-\infty}A_{ij}^+(0)$ is integrable and for every $i\in[1,\cdots,d]$*

$$\mathbb{P}\left(A_{ii}(0)=-\infty\right)=0, \tag{7}$$

then the limit of $\left(\frac{1}{n}y(n,0)\right)$ is given by Equation (3).

If condition (7) is strengthened by demanding that $A_{ii}(0)$ is integrable, then the sequence $\left(\frac{1}{n}x(n,0)\right)$ converges almost-surely[1] *and its limit is given by Equation (4).*

To end this section, we give two examples that show that neither the fixed structure, neither the independence ensure the strong law of large numbers.

Example 1 *([Mai95]).* Let us set $\Omega=\{\omega_0,\omega_1\}$ and $\mathbb{P}=\frac{1}{2}(\delta_{\omega_0}+\delta_{\omega_1})$. Let θ be the function that exchange ω_0 and ω_1 and A, from Ω to $\mathbb{R}_{\max}^{2\times 2}$ be defined by

$$A(\omega_0)=\begin{pmatrix}-\infty & 0\\ 0 & -\infty\end{pmatrix}\text{ and }A(\omega_1)=\begin{pmatrix}-\infty & 1\\ 0 & -\infty\end{pmatrix}.$$

It can be checked that $(\Omega,\theta,\mathbb{P})$ is an ergodic dynamical system, and that the sequence $(A(n))_{n\in\mathbb{N}}$ has fixed structure. Moreover $\mathcal{G}(A)$ is strongly connected. The sequence is a degenerate Markov chain: $A\circ\theta^n=A(\omega_0)\Leftrightarrow A\circ\theta^{n+1}=A(\omega_1)$.

The multiplication by $A(\omega_0)$ exchange the coordinates, and the multiplication by $A(\omega_1)$ does the same, and then increases the first coordinate by 1. Therefore the sequence is defined by the following equations:

$$\begin{array}{rclcrcl} x(2n,z)(\omega_0) &=& (z_1+n,z_2)' & \text{et} & x(2n+1,z)(\omega_0) &=& (z_2,z_1+n)'\\ x(2n,z)(\omega_1) &=& (z_1,z_2+n)' & \text{et} & x(2n+1,z)(\omega_0) &=& (z_2+n+1,z_1)'. \end{array} \tag{8}$$

[1]Actually, the convergence of $\left(\frac{1}{n}x(n,0)\right)$ is always proved under the – slightly weaker but much more difficult to check – condition $\forall m, A^{\{m\}}(0)0\in\mathbb{L}^1$, which appears in Theorem 2.4.

Therefore the sequence $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ almost-surely does not converges.

As $A(n)$ has fixed structure and $\mathcal{G}(A)$ has only one component, it proves that Hypothesis 2. of Theorem 2.5 is necessary.

Example 2 *([BM]).* Let $(A(n))_{n\in\mathbb{N}}$ be the sequence of i.i.d. random variables taking values

$$B = \begin{pmatrix} 0 & -\infty & -\infty \\ 0 & -\infty & -\infty \\ 0 & 1 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & -\infty & -\infty \\ 0 & -\infty & 0 \\ 0 & 0 & -\infty \end{pmatrix}$$

with probabilities $p > 0$ and $1 - p > 0$. Let us compute the action of B and C on vectors of type $(0, x, y)'$, with $x, y \geq 0$:

$$B(0,x,y)' = (0,0,\max(x,y)+1)' \text{ and } C(0,x,y)' = (0,y,x)'.$$

Therefore $x_1(n,0) = 0$ and $\max_i x_i(n+1,0) = \#\{0 \leq k \leq n | A(k) = B\}$. Practically, if $A(n) = B$, then $x(n+1,0) = (0,0,\#\{0 \leq k \leq n|A(k) = B\})'$, and if $A(n) = C$ and $A(n-1) = B$, then $x(n+1,0) = (0,\#\{0 \leq k \leq n|A(k) = B\},0)'$. Since $\left(\frac{1}{n}\#\{0 \leq k \leq n|A(k) = B\}\right)_{n\in\mathbb{N}}$ converges almost-surely to p, we see:

$$\begin{gathered} \lim_n \frac{1}{n}x_1(n,0) = 0 \text{ a.s.} \\ \forall i \in \{2,3\}, \liminf_n \frac{1}{n}x_i(n,0) = 0 \text{ and } \limsup_n \frac{1}{n}x_i(n,0) = p \text{ a.s.}. \end{gathered} \tag{9}$$

Therefore the sequence $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ almost-surely does not converge.

We notice that $\mathcal{G}(A)$ has two components $c_1 = \{1\}$ and $c_2 = \{2,3\}$, with Lyapunov exponents $\gamma^{(1)} = 0$ and $\gamma^{(2)} = p$, and $2 \to 1$. Therefore we check that $A^{\{2\}}(n)$ has a line of $-\infty$ with probability p.

The last example shows the necessity of the integrability conditions in the former theorems: it satisfy every hypothesis of each the three theorems, except for the integrability conditions, but the associated $(x(n,0))_{n\in\mathbb{N}}$ does not satisfy a strong law of large numbers.

Example 3 *(Integrability).* Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real variables satisfying $X_n \geq 1$ a.s. and $\mathbb{E}(X_n) = +\infty$. The sequence of matrices is defined by:

$$A(n) = \begin{pmatrix} -X_n & -X_n & 0 \\ -\infty & 0 & 0 \\ -\infty & -\infty & 0 \end{pmatrix}$$

A straightforward computation shows that $x(n,0) = (\max(-X_n,-n),0,-n)'$ and $y(n,0) = (\max(-X_0,-n),0,-n)'$. Since $\mathbb{P}\left(\lim_n \frac{1}{n}X_n = 0\right) = 0$, it implies that

$\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ converges to $(0,0,-1)'$ in probability but not almost-surely.

Let us notice that the limit of $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$ can also be found by applying one of the theorems and computing that each component has exactly one node and $\gamma^{(1)} = -\mathbb{E}(X_n) = -\infty$, $\gamma^{(2)} = 0$ and $\gamma^{(3)} = -1$.

3 Proofs

3.1 Necessary conditions

3.1.1 Formula for the limit

Let us denote by L the limit of $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$, which exists, according to [BM03], and is assumed to be deterministic.

By definition of $\mathcal{G}(A)$, if (i,j) is an arc of $\mathcal{G}(A)$, then, with positive probability, we have:

$$L_i = \lim_n \frac{1}{n} y_i(n,0) \geq \lim_n \frac{1}{n}\left(A_{ij}(-1) + y_j(n,0)\circ\theta^{-1}\right) = 0 + L_j \circ \theta^{-1} = L_j.$$

If $m \to p$, then for every $i \in c_m$ and $j \in c_p$, there exists a path on $\mathcal{G}(A)$ from i to j, therefore $L_i \geq L_j$. Since this is true for every $j \in F_m$, we have:

$$L_i = \max_{j\in F_m} L_j \tag{10}$$

To show that $\max_{j\in F_m} L_j = \gamma^{[m]}$, we have to study the Lyapunov exponents of sub-matrices. The following proposition states some easy consequences of definition 2.2, which will be useful in the sequel.

Proposition 3.1. *The notations are those of definition 2.2.*

i) for every $m \in [1,\cdots,K]$, $x^{[m]}(n,x_0) = x^{F_m}(n,x_0)$.

ii) for every $m \in [1,\cdots,K]$, and every $i \in c_m$, we have:

$$x_i^{c_m}(n,0) = x_i^{[m]}(n,0) \geq x_i^{\{m\}}(n,0) \geq x_i^{(m)}(n,0).$$

$$y_i^{c_m}(n,0) = y_i^{[m]}(n,0) \geq y_i^{\{m\}}(n,0) \geq y_i^{(m)}(n,0). \tag{11}$$

iii) relation $\to$ is a partial order. Initial and final components are minimal and maximal elements for this order.

iv) If $A(0)$ has no line of $-\infty$, then for every $m \in [1,\cdots,K]$, $A^{[m]}(0)$ has no line of $-\infty$. Practically final components has no line of $-\infty$ and are never trivial.

v) for every $l \in E_m$, we have $\gamma^{(l)} \leq \gamma^{[l]} \leq \gamma^{[m]}$ and $G_m = \{l \in E_m | \gamma^{[l]} = \gamma^{[m]}\}$.

The next result is about Lyapunov exponents. It is already in [BL92] and its proof does not uses the additional hypotheses of this article. For a point by point checking, the reader is referred to [Mer05].

Theorem 3.2 ([Bac92]). *If $(A(n))_{n\in\mathbb{N}}$ is a stationary and ergodic sequence of random matrices with values in $\mathbb{R}_{\max}^{d\times d}$ such that $\max_{i,j} A_{ij}^{+}$ is integrable, then $\gamma(A) = \max_l \gamma^{(l)}$.*

Applying this theorem to sequences $\left(A^{[m]}(n)\right)_{n\in\mathbb{N}}$ and $\left(A^{\{m\}}(n)\right)_{n\in\mathbb{N}}$, we obtain the following proposition

Proposition 3.3. *For every $m \in [1, \cdots, K]$, we have $\gamma(A^{\{m\}}) = \gamma(A^{[m]}) = \gamma^{[m]}$.*

It follows from proposition 3.1 and the definition of Lyapunov exponents that for every component c_m of $\mathcal{G}(A)$,

$$\max_{i\in F_m} L_i = \lim_n \frac{1}{n} \max_{i\in F_m} y_i(n,0) = \gamma(A^{\{m\}}).$$

Combining this with Equation (10) and proposition 3.3, we deduce that the limit of $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$ is given by Equation (3).

3.1.2 $A^{\{m\}}(0)$ has no line of $-\infty$

We still have to show that for every component c_m, $A^{\{m\}}(0)$ has no line of $-\infty$. Let us assume it has one. Therefore, there exists $m \in [1, \cdots, d]$ and $i \in c_m$ such that the set

$$\{\forall j \in H_m, A_{ij}(-1) = -\infty\}$$

has positive probability. On this set, we have:

$$y_i(n,0) \leq \max_{j\in F_m\setminus H_m} A_{ij}(-1) + \max_{j\in F_m\setminus H_m} y_j(n-1,0)\circ\theta^{-1}.$$

Dividing by n and letting n to $+\infty$, we have $L_i \leq \max_{j\in F_m\setminus H_m} L_j$, which, because of Equation (3) becomes $\gamma^{[m]} \leq \max_{k\in E_m\setminus G_m} \gamma^{[k]}$. This last inequality contradicts proposition 3.1 v). Therefore the hypothesis that $A^{\{m\}}(0)$ has a line of $-\infty$.

3.1.3 The limit is constant

Let us assume that $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$ converges almost-surely to a limit L'. Up to a change of probability space, we can assume that $A(n) = A\circ\theta^n$, where A is a random variable and $(\Omega, \theta, \mathbb{P})$ is an invertible ergodic measurable dynamical system.

It follows from [BM03] that $\left(\frac{1}{n}y(n,0)\right)_{n\in\mathbb{N}}$ converges almost-surely and

$$\frac{1}{n}y(n,0)-\frac{1}{n+1}y(n+1,0)\xrightarrow{\mathbb{P}}0.$$

We compound each term of this relation by θ^{n+1} and, since $x(n,0)=y(n,0)\circ\theta^n$, it proves that:

$$\frac{1}{n}x(n,0)\circ\theta-\frac{1}{n+1}x(n+1,0)\xrightarrow{\mathbb{P}}0.$$

When n tends to $+\infty$, it becomes $L'\circ\theta-L'=0$. Since θ is ergodic, this implies that L' is constant.

Since $\frac{1}{n}y(n,0)=\frac{1}{n}x(n,0)\circ\theta^n$, L' and L have the same law. Since L' is constant, $L=L'$ almost-surely, therefore L is also the limit of $\left(\frac{1}{n}x(n,0)\right)_{n\in\mathbb{N}}$. This proves formula (4) and concludes the proof of theorem 2.3 .

3.2 Main theorem

3.2.1 Right products

In this section, we prove Theorem 2.4. We begin with the result on $y(n,0)$.

It follows from propositions 3.1 and 3.3 and the definition of Lyapunov exponents that we have, for every component c_m of $\mathcal{G}(A)$,

$$\limsup_n \frac{1}{n}y^{c_m}(n,0)\leq\gamma^{[m]}\mathbf{1}\text{ a.s.}\,. \tag{12}$$

Therefore, it is sufficient to show that $\liminf_n \frac{1}{n}y^{c_m}(n,0)\geq\gamma^{[m]}\mathbf{1}$ a.s.. Because of proposition 3.1 i), it is sufficient to show that

$$\lim_n \frac{1}{n}y^{\{m\}}(n,0)=\gamma^{[m]}\mathbf{1}. \tag{13}$$

We prove Equation (13) by induction on the size of G_m. The initialization of the induction is exactly Hypothesis 2. of Theorem 2.4.

Let us assume that Equation (13) is satisfied by every m such that the size of G_m is less than N, and let m be such that the size of G_m is $N+1$. Let us take $I=c_m$ and $J=H_m\backslash c_m$. If c_m is not trivial, it is the situation of Hypothesis 3. with $\tilde{A}=A^{\{m\}}$, which has no line of $-\infty$ thanks to Hypothesis 1.. Therefore Equation (6) is satisfied. If c_m is trivial, $\mathcal{G}(B)$ is not strongly connected, but Equation (6) is still satisfied because $D(-1)0=(\tilde{A}(-1)0)^I\in\mathbb{R}^I$.

Moreover J is the union of the c_k such that $k\in G_m\backslash\{m\}$, thus the induction hypothesis implies that:

$$\forall j\in J, j\in c_k\Rightarrow\lim_n\frac{1}{n}(C^{-n}0)_j=\lim_n\frac{1}{n}y_j^{\{k\}}(n,0)=\gamma^{[k]}\text{ a.s..}$$

Because of proposition 3.3 *ii*), $\gamma^{[k]} = \gamma^{[m]}$, therefore the right side of the last equation is $\gamma^{[m]}$ and we have:

$$\lim_n \frac{1}{n}(y^{\{m\}})^J(n,0) = \lim_n \frac{1}{n}C^{-n}0 = \gamma^{[m]}\mathbf{1} \text{ a.s..} \tag{14}$$

Now Equation (6) ensures that for every $i \in I$, there exists almost-surely a $T \in \mathbb{N}$ and a $j \in J$ such that $(B(-1)\cdots B(-T)D(-T-1))_{ij} \neq -\infty$. Since we have $\lim_n \frac{1}{n}(C(-T)\cdots C(-n)0)_j = \gamma^{[m]}$ a.s., it implies that:

$$\begin{aligned}
&\liminf_n \frac{1}{n} y_i^{\{m\}}(n,0) \\
\geq\ & \lim_n \frac{1}{n}(B(-1)\cdots B(-T)D(-T-1))_{ij} + \lim_n \frac{1}{n}(C(-T)\cdots C(-n)0)_j = \gamma^{[m]} \text{ a.s.}
\end{aligned}$$

Because of upper bound (12) and inequality (11), it implies that

$$\lim_n \frac{1}{n}(y^{\{m\}})^I(n,0) = \gamma^{[m]}\mathbf{1} \text{ a.s..,}$$

which, because of Equation (14), proves Equation (13). This concludes the induction and the proof of the result on $y(n,0)$.

3.2.2 Left products

To deduce the results on $x(n,0)$ from those on $y(n,0)$, we introduce the following theorem-definition, which is a special case of the main theorem of J. M. Vincent [Vin97] and directly follows from Kingman's theorem:

Theorem-Definition 3.4 ([Vin97]). *If $(A(n))_{n\in\mathbb{N}}$ is a stationary and ergodic sequence of random matrices with values in $\mathbb{R}_{\max}^{d\times d}$ such that $A(0)0$ is integrable, then there are two real numbers $\gamma(A)$ and $\gamma_b(A)$ such that*

$$\lim_n \frac{1}{n}\max_i x_i(n,0) = \frac{1}{n}\max_i y_i(n,0) = \gamma(A) \text{ a.s.}$$

$$\lim_n \frac{1}{n}\min_i x_i(n,0) = \frac{1}{n}\min_i y_i(n,0) = \gamma_b(A) \text{ a.s.}$$

It implies the following corollary, which makes the link between the results on $y(n,0)$ and those on $x(n,0)$ when all $\gamma^{[m]}$ are equal, that is when $\gamma(A) = \gamma_b(A)$.

Corollary 3.5. *If $(A(n))_{n\in\mathbb{N}}$ is a stationary and ergodic sequence of random matrices with values in $\mathbb{R}_{\max}^{d\times d}$ such that $A(0)0$ is integrable then*

$$\lim_n \frac{1}{n}x(n,0) = \gamma(A)\mathbf{1} \textit{ if and only if } \lim_n \frac{1}{n}y(n,0) = \gamma(A)\mathbf{1}.$$

Let us go back to the proof of the general result on $x(n,0)$. Because of propositions 3.1 and 3.3 and the definition of Lyapunov exponents, we already have, for every component c_m of $\mathcal{G}(A)$,

$$\limsup_n \frac{1}{n} x^{c_m}(n,0) \leq \gamma^{[m]}\mathbf{1} \text{ a.s.}.$$

Therefore it is sufficient to show that $\liminf_n \frac{1}{n} x^{c_m}(n,0) \geq \gamma^{[m]}\mathbf{1}$ a.s.. and even that

$$\lim_n \frac{1}{n} x^{\{m\}}(n,0) = \gamma^{[m]}\mathbf{1}.$$

Because of corollary 3.5, it is equivalent to $\lim_n \frac{1}{n} y^{\{m\}}(n,0) = \gamma^{[m]}\mathbf{1}$. Since all components of $\mathcal{G}(A^{\{m\}})$ are components of $\mathcal{G}(A)$ and have the same Lyapunov exponent $\gamma^{[m]}$, it follows from the result on the $y(n,0)$ applied to $A^{\{m\}}$.

3.3 Independent case

In this section, we prove Theorem 2.5.

Because of Theorem 2.3, it is sufficient to show that, if $(A(n))_{n\in\mathbb{N}}$ is a sequence of i.i.d. random matrices with values in $\mathbb{R}_{\max}^{d\times d}$ such that $\max_{A_{ij}(0)\neq-\infty} |A_{ij}(0)|$ is integrable and for every component c_m, $A^{\{m\}}$ has no line of $-\infty$, then the sequence $\left(\frac{1}{n}x(n,0)\right)$ converges almost-surely. To do this, we will prove that in this situation, the hypotheses of Theorem 2.4 are satisfied. Hypothesis 1. is exactly Hypothesis 1. of Theorem 2.5 and hypotheses 2. and 3. readily follow from the next theorem and lemma respectively.

Theorem 3.6 (D. Hong [Hon01]). *If $(A(n))_{n\in\mathbb{N}}$ is a sequence of i.i.d. random matrices with values in $\mathbb{R}_{\max}^{d\times d}$ such that $A(1)0$ is integrable, $A(1)$ has no line of $-\infty$ and $\mathcal{G}(A)$ is strongly connected, then $\gamma(A) = \gamma_b(A)$.*

Lemma 3.7. *Let $(A(n))_{n\in\mathbb{N}}$ be a stationary sequence of random matrices with values in $\mathbb{R}_{\max}^{d\times d}$ with no line of $-\infty$. Let us assume that there exists a partition (I,J) of $[1,\cdots,d]$ such that $A = \tilde{A}$ satisfy Equation (5), with $\mathcal{G}(B)$ strongly connected. For every $i \in I$, let us define*

$$\mathcal{A}_i := \left\{\forall n \in \mathbb{N}, (B(1)\cdots B(n)D(n+1)0)_i = -\infty\right\}.$$

1. *If $\omega \in \mathcal{A}_i$, then we have $\forall n \in \mathbb{N}, \exists i_n \in I\ (B(1)\cdots B(n))_{ii_n} \neq -\infty$.*

2. *If the random matrices $A(n)$ are i.i.d., and if $\mathbb{P}\left(D = (-\infty)^{I\times J}\right) < 1$, then for every $i \in I$, we have $\mathbb{P}(\mathcal{A}_i) = 0$.*

Proof.

1. For every $\omega \in \mathcal{A}_i$, we prove our result by induction on n.

   Since the $A(n)$ have no line of $-\infty$, there exists an $i_1 \in [1, \cdots, d]$, such that $A_{ii_1}(1) \neq -\infty$. Since $(D(1)0)_i = -\infty$, every entry on line i of $D(1)$ is $-\infty$, that is $A_{ij}(1) = -\infty$ for every $j \in J$, therefore $i_1 \in I$ and $B_{ii_1}(1) = A_{ii_1}(1) \neq -\infty$.

   Let us assume that the sequence is defined up to rank n. Since $A(n+1)$ has no line of $-\infty$, there exists an $i_{n+1} \in [1, \cdots, d]$, such that $A_{i_n i_{n+1}}(n+1) \neq -\infty$.

   Since $\omega \in \mathcal{A}_i$, we have:

   $$-\infty = (B(1) \cdots B(n) D(n+1)0)_i \geq (B(1) \cdots B(n))_{ii_n} + (D(n+1)0)_{i_n} ,$$

   therefore $(D(n+1)0)_{i_n} = -\infty$.
   It means that every entry on line i_n of $D(n+1)$ is $-\infty$, that is $A_{i_n j}(n+1) = -\infty$ for every $j \in J$, therefore $i_{n+1} \in I$ and $B_{i_n i_{n+1}}(n+1) = A_{i_n i_{n+1}}(n+1) \neq -\infty$.

   Finally, we have:

   $$(B(1) \cdots B(n+1))_{ii_{n+1}} \geq (B(1) \cdots B(n))_{ii_n} + B_{i_n i_{n+1}}(n+1) \neq -\infty.$$

2. To every matrix $A \in \mathbb{R}_{\max}^{d \times d}$, we associate the matrix $\widehat{A}$ defined by $\widehat{A}_{ij} = -\infty$ if $A_{ij} = -\infty$ and $A_{ij} = 0$ otherwise. For every matrix $A, B \in \mathbb{R}_{\max}^{d \times d}$, we have $\widehat{AB} = \widehat{A}\widehat{B}$.

   The sequence defined by $R(n) := \widehat{B(1) \cdots B(n)}$ is a Markov chain whose space of states is $\{0, -\infty\}^{I \times I}$ and whose transitions are defined by:

   $$\mathbb{P}\left(R(n+1) = F | R(n) = E\right) = \mathbb{P}\left(\widehat{EB(1)} = F\right).$$

   For every $i, j \in I$, we have $R_{ij}(n) = 0$ if and only if $(B(1) \cdots B(n))_{ij} \neq -\infty$.

   Let E be a recurrent state of this chain. Let us assume there exists a $k \in [1, \cdots, d]$ such that $E_{ik} = 0$. Then, since $\mathcal{G}(B)$ is strongly connected, for every $j \in I$, there exists a $p \in \mathbb{N}$, such that $(B(1) \cdots B(p))_{kj} \neq -\infty$ with positive probability, therefore there exists a state F of the chain, reachable from state E and such that $F_{ij} = 0$. Since E is recurrent, so is F and E and F are in the same recurrence class.

   Let us chose $(i, l) \in I^2$ for a while. In each recurrence class of the Markov chain, either there exists a matrix F such that $F_{il} = 0$, or every matrix has only $-\infty$ on line i.

Now, let us chose $(l,j) \in I \times J$, such that $\mathbb{P}(A_{lj}(1) \neq -\infty) > 0$. Let $\mathcal{E}$ be a set with exactly one matrix F in each recurrence class, such that $F_{il} = 0$ whenever there is such a matrix in the class. Let S_n be the n^{th} time $(R(m))_{m\in\mathbb{N}}$ is in $\mathcal{E}$.

Since the Markov chain has finitely many states and $\mathcal{E}$ intersect every recurrence class, S_n is almost-surely finite. By definition of $\mathcal{E}$, we have almost-surely either $(B(1)\cdots B(S_n))_{il} \neq -\infty$ or $\forall m \in I, (B(1)\cdots B(S_n))_{im} = -\infty$.

It follows from $i)$ that, if $\omega \in \mathcal{A}_i$, we are in the first situation. Therefore, we have, for every $N \in \mathbb{N}$:

$$\mathbb{P}\left[\mathcal{A}_i\right] \leq \mathbb{P}\left[\forall n \in [1,\cdots,N], (D(S_n+1)0)_l = -\infty\right]. \tag{15}$$

Conditioning the event $\{\forall n \in [1,\cdots,N], (D(S_n+1)0)_l = -\infty\}$ by S_N, we have

$$\begin{aligned}
&\mathbb{P}\left[\forall n \in [1,\cdots,N], (D(S_n+1)0)_l = -\infty\right] \\
&\quad= \sum_{k\in\mathbb{N}} \mathbb{P}\left[S_N = k, \forall n \in [1,\cdots,N], (D(S_n+1)0)_l = -\infty\right] \\
&\quad= \sum_{k\in\mathbb{N}} \mathbb{P}\left[S_N = k, (D(k+1)0)_l = -\infty, \forall n \in [1,\cdots,N-1], (D(S_n+1)0)_l = -\infty\right] \\
&\quad= \sum_{k\in\mathbb{N}} \mathbb{P}\left[(D(k+1)0)_l = -\infty\right] \mathbb{P}\left[S_N = k, \forall n \in [1,\cdots,N-1], (D(S_n+1)0)_l = -\infty\right] \\
&\quad= \mathbb{P}\left[(D(1)0)_l = -\infty\right] \mathbb{P}\left[\forall n \in [1,\cdots,N-1], (D(S_n+1)0)_l = -\infty\right],
\end{aligned}$$

because $\{\omega \in \Omega \,|\, S_N = k, \forall n \in [1,\cdots,N-1], (D(S_n+1)0)_l = -\infty\}$ only depends on random matrices $A(1),\cdots,A(k)$.

Finally, we have, for every $N \in \mathbb{N}$:

$$\mathbb{P}\left[\forall n \in [1,\cdots,N], (D(S_n+1)0)_l = -\infty\right] = \left(\mathbb{P}\left[(D(1)0)_l = -\infty\right]\right)^N.$$

Because of the choice of l, $\mathbb{P}\left[(D(1)0)_l = -\infty\right] \leq \mathbb{P}\left[A_{lj}(1)0 = -\infty\right] < 1$ and because of Equation (15), $\mathbb{P}\left(\mathcal{A}_i\right) = 0$.

□

3.4 Fixed structure case

Under the assumptions of Theorem 2.6, the hypotheses of Theorem 2.4 readily follows from the next two lemmas. Therefore Theorem 2.6 is a consequence of Theorem 2.4.

Lemma 3.8. *Let $(\Omega, \theta, \mathbb{P})$ be a measurable dynamical system such that for every $k \in [1, \cdots, d]$, θ^k is ergodic, and let $A : \Omega \to \mathbb{R}_{\max}^{d\times d}$ be a random matrix with no line of $-\infty$, such that $\max_{ij} A_{ij}^{+}(0)$ is integrable. If A has fixed structure and $\mathcal{G}(A)$ is strongly connected, then the $y(n,0)$ associated to $A(n) = A \circ \theta^n$ satisfy*

$$\lim_n \frac{1}{n} y(n,0) = \gamma(A)\mathbf{1} \text{ a.s.}.$$

Lemma 3.9. *If a random matrix A has fixed structure and has no line of $-\infty$, then for every component c_l of $\mathcal{G}(A)$, the random matrix $A^{\{l\}}$ has no line of $-\infty$.*

Proof of Lemma 3.9. Let c_l be a component of $\mathcal{G}(A)$ and let us chose every $i \in H_l$.

If i is in a component c_k of $\mathcal{G}(A)$, then there exists a path from i to a component c_m of $\mathcal{G}(A)$ such that $\gamma^{(m)} = \gamma^{[k]}$. Let j be the first node after i on this path. Let c_p be the component of j. Since $k \to p \to m$, we have:

$$\gamma^{[k]} \geq \gamma^{[p]} \geq \gamma^{[m]} \geq \gamma^{(m)} = \gamma^{[k]}$$

and finally $\gamma^{[p]} = \gamma^{[k]}$. Because of proposition 3.1 v), $\gamma^{[k]} = \gamma^{[l]}$, therefore $\gamma^{[p]} = \gamma^{[l]}$, that is $p \in G_l$, and $j \in H_l$.

By definition of $\mathcal{G}(A)$, we have $\mathbb{P}(A_{ij} \neq -\infty) > 0$, but because of fixed structure, it means $\mathbb{P}(A_{ij} \neq -\infty) = 1$. Therefore $A^{\{l\}}$ has no line of $-\infty$. □

We end this section with the proof of Lemma 3.8 .

Proof of Lemma 3.8. Let L be the limit of $\frac{1}{n} y(n,0)$, which exists according to [BM03].

Because of $\mathcal{G}(A)$'s strong connectivity and the fixed structure, for every entries i,j, there exists $k_{ij} \in [1, \cdots, d]$, such that:

$$\left(A(-1)\cdots A(-k_{ij})\right)_{ij} \neq -\infty \text{ a.s.}.$$

It implies that

$$L_i \geq L_j \circ \theta^{-k_{ij}} \text{ a.s.}. \tag{16}$$

Practically, for $i = j$, it implies $L_i \geq L_i \circ \theta^{k_{ii}}$ almost-surely. Therefore $L_i = L_i \circ \theta^{k_{ii}}$ a.s., and because of the ergodicity of $\theta^{k_{ii}}$, L_i is almost-surely constant.

Equation (16) therefore becomes $L_i \geq L_j$, and by symmetry $L_i = L_j$. Finally we have for every $i \in [1, d]$:

$$L_i = \max_j L_j = \lim_n \frac{1}{n} \max_j y_j(n,0) = \gamma(A) \text{ a.s.}.$$

□

3.5 Precedence case

In this section, we show that the hypotheses of Theorem 2.7 imply those of Theorem 2.4. Hypothesis 1. is obvious because of the precedence condition. Hypotheses 2. and 3. both follow from the next lemma, whose proof is postponed to the end of the section:

Lemma 3.10. *Let $(A(n))_{n\in\mathbb{N}}$ be a sequence satisfying the hypotheses of Theorem 2.7. If $\mathcal{G}(A)$ is strongly connected, then for every $i \in [1, \cdots, d]$, there exists a random variable N with values in $\mathbb{N}$ such that for every $n \geq N$*

$$\forall j \in [1, \cdots, d], (A(-1)\cdots A(-n))_{ij} \neq -\infty \tag{17}$$

Let us assume the hypotheses of Theorem 2.7. Without loss of generality, we assume that $A(n) = A \circ \theta^n$, where $(\Omega, \theta, \mathbb{P})$ is a measurable dynamical system and A is a random matrix.

To check hypothesis 2., we deduce from the lemma that if $\mathcal{G}(A)$ is strongly connected, then for every $i \in [1, \cdots, d]$,

$$y_i(n, 0) \geq \min_j (A(-1)\cdots A(-N))_{ij} + \max_j y_j(n - N, 0) \circ \theta^{-N},$$

and therefore

$$\liminf_n \frac{1}{n} y_i(n, 0) \geq \lim_n \frac{1}{n} \min_j (A(-1)\cdots A(-N))_{ij} + \lim_n \frac{1}{n} \max_j y_j(n-N, 0) \circ \theta^{-N} = \gamma(A).$$

Because of the definition of $\gamma(A)$, we also have $\limsup_n \frac{1}{n} y_i(n, 0) \leq \gamma(A)$, therefore $\lim_n \frac{1}{n} y_i(n, 0) = \gamma(A)$.

We apply this result to $A^{(m)}$ where c_m is a dominating component, and this proves that Hypothesis 2. is satisfied.

To check Hypothesis 3., we apply Lemma 3.10 to matrix B of decomposition (5), and we conclude the proof thanks to the ergodicity, that ensures there exists $n \geq N$ such that $D_{i'j}(-n-1) \neq -\infty$, provided $\mathbb{P}(D_{i'j}(1) \neq -\infty)$. Since there is such a pair (i', j), it proves:

$$(B(-1)\cdots B(-n)D(-n-1)0)_i \geq (B(-1)\cdots B(-n))_{ii'} + D_{i'j}(-n-1) > -\infty,$$

that is Hypothesis 3. is checked.

Proof of Lemma 3.10. Because of the ergodicity of θ, if $\mathbb{P}(A_{ij}(0) \neq -\infty) > 0$, then there exists almost-surely an n_{ij} such that $A_{ij}(n_{ij}) \neq -\infty$. That being the case, Poincaré recurrence theorem states that there are infinitely many such n_{ij}.

Let us chose $i \in [1, d]$. Because of the precedence condition, the sequence of sets

$$\mathcal{A}(n) = \left\{ j \,\middle|\, (A(-1)\cdots A(-n))_{ij} \neq -\infty \right\}$$

is increasing with n. Let us show that, from some rank $\mathcal{A}(n) = [1, d]$.

Since $\mathcal{G}(A)$ is strongly connected, there exists for every j a finite sequence $i_0 = i,\ i_1, \cdots, i_k = j$ such that for every $l \in [1, k]$, $\mathbb{P}\left(A_{i_l i_{l+1}}(0) \neq -\infty\right) > 0$. Because of what we said in the first paragraph of the proof, there are almost-surely $n_1 < \cdots < n_k$ such that $A_{i_l i_{l+1}}(n_l) \neq -\infty$. Then, for every $n \geq n_i$, $(A(-1)\cdots A(-n))_{ii_{l+1}} \neq -\infty$, and for every $n \geq n_k$, we have $(A(-1)\cdots A(-n))_{ij} \neq -\infty$, that is $j \in \mathcal{A}(n)$. Since it is true for every j, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $\mathcal{A}(n) = [1, \cdots, d]$, which concludes the proof of the lemma.

4 Acknowledgements

I have done this work both during my PhD at Université de Rennes 1 and as a JSPS postdoctoral fellow at Keio University. During this time, many exchange with J. Mairesse have been a great help. This paper owes much to him. □

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